Department of Mathematics
Brooklyn College

Final Examination— Spring 2005
Mathematics 11.1 Section TF9

Show all work.

1*a*) Define what it means for a number c to be an upper bound of the subset S of $\mathbb{R}$ ($\mathbb{R}$ denotes the set of real numbers).

b) Define what it means for a number c to be the least upper bound of the subset S of $\mathbb{R}$.

c) Let S be a subset of $\mathbb{R}$ and assume that $c = \sup S$. Let $\epsilon > 0$. Show that there is an $x \in S$ such that $x > c - \epsilon$.

2*a*) Give the definition for a set S in a metric space (E, d) being open.

b) Show that if the set S in a metric space (E, d) is not closed then there is a point $p \notin S$ of E and a sequence $\{p_n\}_{n=1}^{\infty}$ of elements of S that converges to p.

3*a*) Let (E, d) be a metric space and let $\{p_n\}_{n=1}^{\infty}$ be a convergent sequence of elements of E. Show the set $\{p_n : n \ge 1\}$ is bounded.

b) Let S be a nonempty closed set of real numbers, and assume that S is bounded from above. Prove that S has a maximum.

4 Let (E, d) and (E', d') be metric spaces. Let $f : E \to E'$ be a function and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : E \to E'$.

a) Give a definition of f being continuous on E. That is, give a definition of the statement that f is continuous at every point $p \in E$.

b) Give a definition of f being uniformly continuous on E. *After giving the definition*, explain what the difference is between continuity and uniform continuity. (The emphasis is on the word *after*; the definition and the explanation must not be intermixed.)

c) Give a definition of the sequence $\{f_n\}_{n=1}^{\infty}$ being convergent to f. That is, give a definition of the statement that $f(p) = \lim_{n\to\infty} f_n(p)$ for every $p \in E$.

d) Give a definition of the sequence $\{f_n\}_{n=1}^{\infty}$ being uniformly convergent to f. *After giving the definition*, explain what the difference is between convergence and uniform convergence. (The emphasis is on the word *after*; the definition and the explanation must not be intermixed.)

5*a*) Let (E_1, d_1), (E_2, d_2), and (E_3, d_3) be metric spaces, and let $g\colon E_1 \to E_2$ and $f\colon E_2 \to E_3$ be functions. Let $p \in E_1$, and assume that g is continuous at p, and f is continuous at $g(p)$. Prove that $f \circ g$ is continuous at p.

b) Let f be a real-valued function on an open subset U of $\mathbb{R}$, and let $c \in U$ be such that f attains a maximum in U at the point c. Assume, further, that $f'(c)$ exists. Prove that $f'(c) = 0$.

6*a*) State the first criterion for a function f on the interval $[a, b]$ to be Riemann integrable (i.e., the criterion that is stated in terms of Riemann sums).

b) Assume that f is a continuous function on the interval $[a, b]$. Show that f is Riemann integrable on $[a, b]$.

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7. *a*) Give a necessary and sufficient condition involving sums of form $\sum_{k=m+1}^{n} a_k$ for the convergence of the series $\sum_{n=1}^{\infty} a_n$ (this is the criterion about the partial sums of a series forming a Cauchy sequence).

b) Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

8. *a*) State the result on the termwise integrability of a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$.

b) According to the result (it is hoped) you correctly stated in Part *a*), the geometric series

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

can be integrated termwise to obtain

$$\log(1+x) = \int_0^x \frac{1}{1+t}\,dt = \sum_{n=0}^{\infty} \int_0^x (-1)^n t^n\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

for $-1 < x < 1$. Termwise integration is not justified in case $x = 1$. Show that, nevertheless, the result is true even in case $x = 1$. That is, we have

$$\log 2 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}.$$

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