Department of Mathematics
Brooklyn College

Final Examination— Fall 2004
Mathematics 37.1 Section TY10

Show all work.

1. *a*) Evaluate $\sqrt{x^2+y^2}$ for $x = 3 \pm 0.02$ and $y = 4 \pm 0.05$.

b) Assume that the differentiable function f has n zeros in the interval (a, b). Explain why f' has at least $n-1$ zeros in the same interval.

2. *a*) Let $P(x)$ be a polynomial of degree at most 2 that agrees with $e^{x/2}$ at the points $x = -1$, $x = 0$, and $x = 2$. Give an estimate for $P(1) - e^{1/2}$. (Hint: use the error formula for interpolation).

b) Find a polynomial

$$P(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^2(x-3)$$

such that $P(1) = 3$, $P'(1) = 2$, $P''(1) = 10$, and $P(3) = 11$. (Hint: you are looking for a Newton-Hermite interpolation polynomial.)

3. *a*) To find the root of the equation $e^x - 4x^2 = 0$ by fixed point iteration in the interval $[4, 5]$, which form of the equation would you use: (i) $x = e^{x/2}/2$ or (ii) $x = \ln 4 + 2\ln x$? Give reasons, and then, using one of these two forms, do one step of fixed point iteration starting with the value $x = 4.2$.

b) Assume $f(\alpha) = \alpha$, and, for some $c > 0$, we have $|f'(x)| < q$ for every x in the interval $(\alpha - c, \alpha + c)$. Show that, we have

$$|f(x) - \alpha| < q|x - \alpha|$$

for any x the above interval. (Hint: use the Mean-Value Theorem).

4. *a*) Using adaptive integration, we want to calculate the integral

$$\int_0^2 \frac{10}{1+100x^4}\,dx,$$

with one decimal precision, i.e., with an absolute error having absolute value not larger than 0.05. Decide whether the interval $(1/2, 5/8)$ needs to be further subdivided during

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(continued)

this calculation. Give reasons for your decision. To help you make your decision, note that, writing

$$f(x) = \frac{10}{1 + 100x^4},$$

we have $f(1/2) = 1.379,310$, $f(9/16) = 0.908,158$, and $f(5/8) = 0.615,051$.

b) Explain why you would prefer to use adaptive integration to Romberg integration when calculating this integral.

5. *a*) Given a differential equation $y' = -y^2$ with the initial condition $y(0) = 1$, do one step of the Taylor series method to calculate $y(.1)$. For this, use the first four terms of the Taylor series (that is, the last term to be used will involve the third derivative of y).

b) Consider the differential equation $y' = f(x, y)$ with initial condition $y(x_0) = y_0$. Show that, with $x_1 = x_0 + h$, the solution at x_1 can be obtained with an error $O(h^3)$ by the formula

$$y_1 = y_0 + hf\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0)\right).$$

In other words, this formula describes a Runge-Kutta method of order 2.

6. *a*) Solve the equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 14 \end{pmatrix}.$$

b) Consider the following system of equations:

$$\begin{aligned} x - 10y + \ \ 3z &= \ \ 6 \\ 10x + \quad y - \quad z &= 10 \\ 2x + \ \ 2y - 10z &= \ \ 4 \end{aligned}$$

Write the equations that can be used to solve this system by Gauss-Seidel iteration; that is, write the equations that express the new values $x^{(k+1)}$, $y^{(k+1)}$, $z^{(k+1)}$ in terms of the old values $x^{(k)}$, $y^{(k)}$, $z^{(k)}$. It may be necessary to change the order of equations for Gauss-Seidel iteration to be successful. If so, do not forget to do this.